Preconditioners for Eigenvalue Problems Arising in Linear Stability Analysis

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This work concerns computational methods for solving generalized eigenvalue problems of the type that arise in stability analysis of models of incompressible flow. *Linear stability analysis* of the incompressible Navier-Stokes equations leads to eigenvalue problems of the form

$$\begin{pmatrix} \mathcal{L}\mathbf{v} + \text{grad } p \\ -\text{div } \mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathcal{M}\mathbf{v} \\ 0 \end{pmatrix}, \text{ where } \begin{cases} \mathcal{L}\mathbf{v} = -\nu\nabla^2\mathbf{v} + (\mathbf{u}\cdot\text{grad})\mathbf{v} + (\mathbf{v}\cdot\text{grad})\mathbf{u}, \\ \mathcal{M}\mathbf{v} = -\mathbf{v}. \end{cases} \tag{1}$$

Discretization of (1) yields a real algebraic eigenvalue problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix} = \lambda \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix}, \quad \text{or} \quad \mathcal{A}x = \lambda\mathcal{B}x. \tag{2}$$

Efficient eigenvalue algorithms require the solution of linear systems in which the coefficient matrix is (a possibly shifted version of) the matrix on the left of (2). The key is to combine good iterative eigenvalue solvers with effective preconditioners for $\mathcal{A}$ or $\mathcal{A} - \theta_k\mathcal{B}$.

Our approach to preconditioning uses a technique called *tuning*, developed by Freitag and Spence [1], which enhances the effect of preconditioners designed for $\mathcal{A}_0 = \mathcal{A}$ in the setting of eigenvalues. Suppose inverse iteration is used for (2), leading to a sequence of systems of the form

$$(\mathcal{A} - \theta_k\mathcal{B})y^{(k)} = \mathcal{B}x^{(k)}. \tag{3}$$

Given a preconditioner $\mathcal{Q}_\mathcal{A}$ for $\mathcal{A}$, let $\hat{\mathcal{Q}}_\mathcal{A} = \mathcal{Q}_\mathcal{A} + (w^{(k)} - \mathcal{Q}_\mathcal{A}x^{(k)})x^{(k)T}$, where $w^{(k)}$ is a "tuning" vector to be specified. If $w^{(k)} = \mathcal{A}x^{(k)}$, then $\mathcal{B}x^{(k)}$ tends to an approximate eigenvector of the preconditioned matrix $(\mathcal{A} - \theta\mathcal{B})\hat{\mathcal{Q}}_\mathcal{A}^{-1}$. That is, as the inverse iterate $x^{(k)}$ tends to an eigenvector of (2), the preconditioner used within this computation is adapted so that the right-hand side in the system *simultaneously* converges to an eigenvector of the preconditioned matrix. This enhances the convergence of the linear system solver for (3).

For this idea to useful as a general tool, it must be integrated into a fast eigenvalue method such as the *implicitly restarted Arnoldi* (IRA) method [2]. We demonstrate that this can be done by constructing small "tuning spaces" using the vectors discarded during the IRA iteration. We describe the impact of these ideas on the performance of solution algorithms for stability analysis of incompressible flows, showing that appropriate choices of tuning spaces can reduce by factors of two or higher the number of iterations required for the preconditioned solves used for such analysis.

References

[1] M. A. Freitag and A. Spence, Convergence rates for inexact inverse iteration with application to preconditioned iterative solves *BIT*, 47:27–44, 2007.

[2] D. C. Sorensen, Implicit application of polynomial filters in a k-step Arnoldi method *SIAM J. Matrix. Anal. Anal.*, 13:357-385, 1992.

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